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Raising and Lowering Operators for a Class of Exactly Solvable Quantum Nonlinear Harmonic Oscillators

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Abstract In this paper, we study a new class of exactly solvable quantum nonlinear harmonic oscillators from the viewpoint of the raising and lowering operators. The energy spectrum for the Hamiltonian and the ground state are also given explicitly.

Keywords Raising and lowering operators · Quantum nonlinear harmonic oscillators

1 Introduction

It is well-known that quantum linear harmonic oscillator is one of the exactly solvable models in quantum mechanics. Recently, the quantum nonlinear harmonic oscillator (QNHO) has been studied with great interest [1–6]. For instance, Carinena, Ranada and Santander have proposed a kind of one-dimensional model for the QNHO [2], whose Hamiltonian reads

$$H_{CRS} = \frac{1}{2m}(\mathcal{K}p^2 - i\hbar\lambda xp) + \frac{\alpha^2 x^2}{2(1+\lambda x^2)}, \tag{1}$$

where $\mathcal{K} = 1 + \lambda x^2$, λ is a real number, m is the mass for the particle, $p = -i\hbar\frac{d}{dx}$, $[x, p] = i\hbar$, and $\hbar$ is the Planck constant. The energy spectrum of such a λ-dependent Hamiltonian can be determined by the supersymmetric approach. When $\lambda = 0$, it naturally reduces to the usual quantum linear harmonic oscillator.

In this paper, from the viewpoint of raising and lowering operators, we aim to study a more general class of exactly solvable quantum harmonic oscillators. The Hamiltonian of the general QNHO is given by

$$H = \frac{1}{2m}(\mathcal{K}p^2 - i\hbar\lambda xp) + V(x), \tag{2}$$

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where the potential $V(x)$ is determined by the operator method. The energy spectrum for the Hamiltonian and the ground state will be given explicitly due to the raising and lowering operators.

2 Raising and Lowering Operators for the Hamiltonian

For a Hamiltonian operator H, if there are operators $\hat{\mathcal{L}}^{\pm}$ satisfying the following commutation relation:

$$[H, \hat{\mathcal{L}}^{\pm}] = \hat{\mathcal{L}}^{\pm} g^{\pm}(H), \tag{3}$$

then $\hat{\mathcal{L}}^{+}$ and $\hat{\mathcal{L}}^{-}$ are called the raising and lowering operators of operator H, respectively. When $\hat{\mathcal{L}}^{-} = a$, $\hat{\mathcal{L}}^{+} = a^{\dagger}$, and $g^{\pm}(H) = \pm\hbar\omega$, (3) reduces to the usual case of the quantum linear harmonic oscillator, namely, $[H, a] = -a\hbar\omega$, $[H, a^{\dagger}] = a^{\dagger}\hbar\omega$. The readers who are interested in the general definition of raising and lowering operators may refer to Refs. [7, 8]. Moreover, it is worth mentioning that the explicit form of the raising and lowering operators $\hat{\mathcal{L}}^{\pm}$ for a specific Hamiltonian system need not be mutually adjoint [7].

Now let us come to study the raising and lowering operators for the Hamiltonian operator H as shown in (2). We start from the following definition for the generalized "coordinate" operator:

$$X = \sum_{j=0}^{\ell} a_j x^j, \quad \ell = 1, 2, 3, \ldots, \tag{4}$$

where the coefficients a_j's are some real numbers. When $a_1 = 1$, $a_j = 0$ (for $j \neq 1$), then $X = x$ is the usual coordinate operator.

In the next step, one may define the generalized "momentum" operator through the following commutation relation:

$$[H, X] = -\frac{i\hbar}{m} P_X. \tag{5}$$

By using the basic commutation relation $[x, p] = i\hbar$, from (2) and (5) we obtain that

$$P_X = \mathcal{K} X' p - \frac{1}{2} i\hbar(\mathcal{K} X'' + \lambda x X'). \tag{6}$$

Obviously, if $\lambda = 0$, $X = x$, then $P_X = p$ is nothing but the usual "momentum" operator.

To simplify the calculation, one may let

$$\mathcal{K} X'' + \lambda x X' = AX + B, \tag{7}$$

which yields

$$A = \lambda\ell^2, \qquad B = 2a_2 - Aa_0. \tag{8}$$

Based on which, one has the following commutation relation:

$$[H, P_X] = \left[-\frac{i\hbar}{m}\mathcal{K}(AX + B)\right] p^2 - \frac{\hbar^2}{m}[\lambda x(AX + B) + A\mathcal{K}X'] p + \frac{i\hbar^3}{4m} A(AX + B) + i\hbar\mathcal{K}X'V'(x), \tag{9}$$

or

$$[H, P_X] = X\left[-i\hbar 2AH - \frac{i\hbar^3}{4m}A^2 + i\hbar\Omega\right] + P_X\left(-\frac{\hbar^2}{m}A\right) - i\hbar 2BH + i\hbar\tau, \tag{10}$$

where we have used

$$p^2 = 2mH - V(x) + i\hbar\lambda xp, \tag{11}$$

which can be derived directly from (2), and the assumption relation

$$i\hbar\left[-\frac{\hbar^2}{4m}AB + \mathcal{K}X'V'(x) + 2(AX + B)V(x)\right] = i\hbar(\Omega X + \tau), \tag{12}$$

where Ω and τ are real numbers.

In the following, we shall use the similar approach developed in Ref. [9] to construct the raising and lowering operators. From (5) and (10), we obtain

$$\begin{aligned} HX &= XH - \frac{i\hbar}{m}P_X, \\ HP_X &= X\left(-i\hbar 2AH - \frac{i\hbar^3}{4m}A^2 + i\hbar\Omega\right) + P_X\left(-\frac{\hbar^2}{m}A + H\right) - i\hbar 2BH + i\hbar\tau, \end{aligned} \tag{13}$$

which can be recast into a matrix form as

$$H\left(X, P_X, 1, \frac{1}{H}\right) = \left(X, P_X, 1, \frac{1}{H}\right)G, \tag{14}$$

where

$$G = \begin{pmatrix} H & i\hbar(-2AH - \frac{1}{2}\epsilon A^2 + \Omega) & 0 & 0 \\ -\frac{i\hbar}{m} & H - 2\epsilon A & 0 & 0 \\ 0 & -i\hbar 2BH & H & 0 \\ 0 & i\hbar\tau H & 0 & H \end{pmatrix}, \quad \epsilon = \frac{\hbar^2}{2m}. \tag{15}$$

By solving the equation

$$\det(G - \lambda I) = 0, \tag{16}$$

we obtain the eigenvalues of G as follows:

$$\lambda_1 = H, \qquad \lambda_2 = H, \qquad \lambda_3 = H - A\epsilon - T(H), \qquad \lambda_4 = H - A\epsilon + T(H), \tag{17}$$

where

$$T(H) = \sqrt{\epsilon(2\Omega - 4AH)}. \tag{18}$$

The diagonalized G can be written in the form

$$G = R\Lambda R^{-1}, \tag{19}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \tag{20}$$

and the diagonalizing matrix is

$$R = \begin{pmatrix} 0 & 0 & 2AH + \frac{1}{2}\epsilon A^2 - \Omega & 2AH + \frac{1}{2}\epsilon A^2 - \Omega \\ 0 & 0 & -\frac{i}{\hbar}(A\epsilon + T(H)) & -\frac{i}{\hbar}(A\epsilon - T(H)) \\ 0 & 1 & 2BH & 2BH \\ 1 & 0 & -H\tau & -H\tau \end{pmatrix}. \tag{21}$$

Equation (14) can be rewritten as $H(X, P_X, 1, \frac{1}{H}) = (X, P_X, 1, \frac{1}{H})G = (X, P_X, 1, \frac{1}{H})\times R\Lambda R^{-1}$, or

$$H\left(X, P_X, 1, \frac{1}{H}\right)R = \left(X, P_X, 1, \frac{1}{H}\right)R\Lambda. \tag{22}$$

Let

$$\left(\frac{1}{H}, 1, b, b^+\right) = \left(X, P_X, 1, \frac{1}{H}\right)R, \tag{23}$$

then one arrives at the raising and lowering operators for the Hamiltonian operator H as

$$b = X\left(2AH + \frac{1}{2}\mathcal{E}A^2 - \Omega\right) - P_X\left(\frac{i}{\hbar}\right)(A\mathcal{E} + T(H)) + 2BH - \tau, \tag{24}$$

$$b^+ = X\left(2AH + \frac{1}{2}\mathcal{E}A^2 - \Omega\right) - P_X\left(\frac{i}{\hbar}\right)(A\mathcal{E} - T(H)) + 2BH - \tau. \tag{25}$$

This point can be seen clearly if one substitutes (23) into (22), he obtains

$$H\left(\frac{1}{H}, 1, b, b^+\right) = \left(\frac{1}{H}, 1, b, b^+\right)\Lambda, \tag{26}$$

namely,

$$[H, b] = b(\lambda_3 - H), \qquad [H, b^+] = b^+(\lambda_4 - H), \tag{27}$$

which recovers the definition of raising and lowering operators as shown in (3).

3 A Class of Exactly Solvable Potentials

Equation (12) can be rewritten as

$$V'(x) + \frac{2(AX + B)}{\mathcal{K}X'}V(x) = \frac{1}{\mathcal{K}X'}\left(\Omega X + \frac{1}{2}\epsilon AB + \tau\right), \tag{28}$$

which is the differential equation for determining the potentials $V(x)$.

Let

$$V(x) = \frac{U(x)}{\mathcal{K}(X')^2}, \tag{29}$$

then (28) becomes

$$U'(x) = X'\left(\Omega X + \frac{1}{2}\epsilon AB + \tau\right). \tag{30}$$

From (30), we have

$$U(x)=\frac{1}{2}\Omega X^2+\frac{1}{2}(\epsilon AB+2\tau)X+\frac{1}{2}C, \tag{31}$$

thus, the solutions for the potentials are

$$V(x)=\frac{1}{2\mathcal{K}(X')^2}[\Omega X^2+(\epsilon AB+2\tau)X+C], \tag{32}$$

where C is a real number to be determined later on.

The class of potentials $V(x)$ as shown in (32) are exactly solvable by the method of raising and lowering operators. Now let us come to study the ground state and ground energy, which satisfy the equations:

$$b|\psi_0\rangle=0, \tag{33}$$

$$H|\psi_0\rangle=E_0|\psi_0\rangle, \tag{34}$$

where $|\psi_0\rangle$ is the ground state and E_0 is the ground energy. We may have

$$\frac{d|\psi_0\rangle}{dx}+\frac{1}{\mathcal{K}X'}(\Delta_1+\Delta_2 X)|\psi_0\rangle=0, \tag{35}$$

where

$$\Delta_1=-\frac{2BE_0-\tau}{A\epsilon+T(E_0)}+\frac{1}{2}B, \tag{36}$$

$$\Delta_2=\frac{\Omega-\frac{1}{2}\epsilon A^2-2AE_0}{A\epsilon+T(E_0)}+\frac{1}{2}A. \tag{37}$$

Then the ground state reads

$$|\psi_0\rangle=\mathcal{N}_0\exp\left\{-\int\frac{\Delta_1+\Delta_2 X}{\mathcal{K}X'}dx\right\}, \tag{38}$$

where $\mathcal{N}_0$ is the normalized constant.

Equation (34) leads to

$$H|\psi_0\rangle=\epsilon\Delta_2|\psi_0\rangle+\left[V(x)-\epsilon\cdot\frac{(AX+B)(\Delta_1+\Delta_2X)+(\Delta_1+\Delta_2X)^2}{\mathcal{K}(X')^2}\right]|\psi_0\rangle=E_0|\psi_0\rangle, \tag{39}$$

therefore one has

$$E_0=\epsilon\Delta_2, \tag{40}$$

$$V(x)=\epsilon\cdot\frac{(AX+B)(\Delta_1+\Delta_2X)+(\Delta_1+\Delta_2X)^2}{\mathcal{K}(X')^2}. \tag{41}$$

Equations (32) and (41) are required to be consistent, then we have

$$\begin{aligned}&(\Omega-2\epsilon A\Delta_2-2\epsilon\Delta_2^2)X^2+[\epsilon AB+2\tau-2\epsilon(A\Delta_1+B\Delta_2)-4\epsilon\Delta_1\Delta_2]X\\&\quad+C-2\epsilon(B\Delta_1+\Delta_1^2)=0,\end{aligned} \tag{42}$$

in other words,

$$\Omega - 2\epsilon A\Delta_2 - 2\epsilon\Delta_2^2 = 0, \tag{43}$$

$$\epsilon AB + 2\tau - 2\epsilon(A\Delta_1 + B\Delta_2) - 4\epsilon\Delta_1\Delta_2 = 0, \tag{44}$$

$$C - 2\epsilon(B\Delta_1 + \Delta_1^2) = 0. \tag{45}$$

By solving (40) and (43), we obtain

$$\Omega = \frac{2(E_0^2 + \epsilon A E_0)}{\epsilon}, \tag{46}$$

or the ground state energy as

$$E_0 = \frac{1}{2}\left(-\epsilon A + \sqrt{\epsilon^2 A^2 + 2\epsilon\Omega}\right). \tag{47}$$

From (46) and (18), we have

$$T(E_0) = 2E_0, \tag{48}$$

based on which, one may find that (44) is consistent with (36). And the constant C in $V(x)$ is

$$C = 2\epsilon(B\Delta_1 + \Delta_1^2). \tag{49}$$

The excitation states $|\psi_{n+1}\rangle$ can be constructed by the action of b^+ on $|\psi_n\rangle$

$$|\psi_{n+1}\rangle \propto b^\dagger|\psi_n\rangle, \tag{50}$$

and the corresponding energy E_n can be determined by the lowering operator as follows. Let

$$[H, b] = -bg(H), \tag{51}$$

then one has

$$g(H) = H - \lambda_3 = A\epsilon + T(H). \tag{52}$$

Due to

$$[H, b]|\psi_n\rangle = -g(E_n)b|\psi_n\rangle, \tag{53}$$

we have

$$H(b|\psi_n\rangle) = (E_n - g(E_n))(b|\psi_n\rangle). \tag{54}$$

Let $b|\psi_n\rangle = |\psi_{n-1}\rangle$, then

$$g(E_n) = E_n - E_{n-1}, \tag{55}$$

i.e., the physical meaning of $g(H)$ is an energy interval operator. From (52) and (55), we obtain the energy spectrum as

$$E_n = E_0 - n^2 A\epsilon + nT(E_0) = (2n+1)E_0 - n^2 A\epsilon. \tag{56}$$

Specially, for the quantum linear harmonic oscillator, $\lambda = 0$, $E_0 = \hbar\omega/2$, then the above equation reduces to the well-known formula $E_n = (n + 1/2)\hbar\omega$.

In the following, we would like to provide some explicit examples for the potentials as in 32), which are characterized by the parameter ℓ.

Example 1 For $\ell = 1$, one has

$$X = a_1 x + a_0, \qquad A = \lambda, \qquad B = -a_0\lambda, \tag{57}$$

where $a_1 \neq 0$, and a_0 is an arbitrary real number. The potential $V(x)$ takes the form

$$V(x) = \frac{1}{\mathcal{K}}\left[\frac{\Omega}{2}x^2 + \left(\frac{\Omega a_0}{a_1} + \frac{\epsilon AB + 2\tau}{2a_1}\right)x + \frac{\Omega a_0^2 + (\epsilon AB + 2\tau)a_0 + C}{2a_1^2}\right], \tag{58}$$

and the ground state and ground energy are

$$|\psi_0\rangle = \mathcal{N}_0 \exp\left\{-\frac{\Delta_1}{a_1\sqrt{\lambda}}\arctan(\sqrt{\lambda}x) - \Delta_2\left[\frac{a_0}{a_1\sqrt{\lambda}}\arctan(\sqrt{\lambda}x) + \frac{1}{2\lambda}\ln(1+\lambda x^2)\right]\right\}, \tag{59}$$

$$E_0 = \frac{1}{2}\left(-\lambda\epsilon + \sqrt{\lambda^2\epsilon^2 + 2\epsilon\Omega}\right). \tag{60}$$

For the special case, when $a_0 = 0$, $\tau = 0$ and let $\hbar = m = 1$, $\Omega = \alpha^2$, we have $\Delta_1 = 0$, $B = 0$ and

$$V(x) = \frac{1}{\mathcal{K}}\left(\frac{\alpha^2}{2}x^2\right) = \frac{1}{2}\cdot\frac{\alpha^2 x^2}{1+\lambda x^2}, \tag{61}$$

which is just the QNHO model in [1]. After denoting $\Delta_2 = \beta$, the ground state and the ground energy can be simplified as

$$|\psi_0\rangle = \mathcal{N}_0 \exp\left\{-\Delta_2\frac{1}{2\lambda}\ln(1+\lambda x^2)\right\} = \left(\frac{1}{1+\lambda x^2}\right)^{\frac{\beta}{2\lambda}}, \tag{62}$$

$$E_0 = \frac{1}{2}\Delta_2 = \frac{1}{2}\beta. \tag{63}$$

Therefore, by using the operator method, one can derive some exactly solvable QNHO models, the QNHO model proposed in [1] belongs to the case with $\ell = 1$.

Example 2 For $\ell = 2$, similarly one has

$$X = a_2 x^2 + a_0, \tag{64}$$

where $a_2 = \frac{1}{2}(B + 4\lambda a_0) \neq 0$, a_0 and B are arbitrary real numbers. From (32), we obtain

$$V(x) = \frac{1}{\mathcal{K}x^2}\left(\frac{\Omega}{8}x^4 + \frac{2\Omega a_0 + \epsilon AB + 2\tau}{8a_2}x^2 + \frac{\Omega a_0^2 + (\epsilon AB + 2\tau)a_0 + C}{8a_2^2}\right) \tag{65}$$

and $|\psi_0\rangle = \mathcal{N}_0 \exp\{-\Delta_1 Q_1 - \Delta_2 Q_2\}$, where

$$Q_1 = \frac{1}{2a_2}\ln x - \frac{1}{4a_2}\ln(1+\lambda x^2), \tag{66}$$

$$Q_2 = \frac{a_0}{2a_2}\ln x + \frac{a_2 - a_0\lambda}{4a_2\lambda}\ln(1+\lambda x^2). \tag{67}$$

Example 3 For $\ell = 3$, similarly one has

$$X = a_3 x^3 + a_1 x + a_0, \quad a_3 = \frac{4}{3}\lambda a_1, \tag{68}$$

where $a_0,\ a_1$ are real numbers, $a_1 \neq 0$, The potential is

$$V(x) = \frac{1}{\mathcal{K}(1+4\lambda x^2)^2} \cdot \left[\frac{8}{9}\Omega\lambda^2 x^6 + \frac{4}{3}\Omega\lambda x^4 + \frac{2\lambda}{3a_1}(2\Omega a_0 + \epsilon AB + 2\tau)x^3 + \frac{\Omega}{2}x^2 + \frac{2\Omega a_0 + \epsilon AB + 2\tau}{2a_1}x + \frac{\Omega a_0^2 + (\epsilon AB + 2\tau)a_0 + C}{2a_1^2}\right], \tag{69}$$

and $|\psi_0\rangle = \mathcal{N}_0 \exp\{-\Delta_1 Q_1 - \Delta_2 Q_2\}$, where

$$Q_1 = -\frac{\arctan(\sqrt{\lambda}x) - 2\arctan(2\sqrt{\lambda}x)}{3a_1\sqrt{\lambda}}, \tag{70}$$

$$Q_2 = \frac{1}{18a_1\lambda}\big\{-6a_0\sqrt{\lambda}\arctan(\sqrt{\lambda}x) + 12a_0\sqrt{\lambda}\arctan(2\sqrt{\lambda}x) + a_1[\ln(1+\lambda x^2) + 2\ln(1+4\lambda x^2)]\big\}. \tag{71}$$

Example 4 For $\ell = 4$, similarly one has

$$X = a_4 x^4 + a_2 x^2 + a_0, \quad a_4 = a_2\lambda, \tag{72}$$

where a_0 and a_2 are real numbers, $a_2 \neq 0$. The potential is

$$V(x) = \frac{1}{\mathcal{K}x^2(1+2\lambda x^2)^2} \cdot \left[\frac{\Omega\lambda^2}{4}x^8 + \frac{\Omega\lambda}{2}x^6 + \frac{\Omega(a_2 + 2a_0\lambda) + (\epsilon AB + 2\tau)\lambda}{4a_2}x^4 + \frac{\Omega a_0 + \epsilon AB + 2\tau}{4a_2}x^2 + \frac{\Omega a_0^2 + (\epsilon AB + 2\tau)a_0 + C}{4a_2^2}\right], \tag{73}$$

and $|\psi_0\rangle = \mathcal{N}_0 \exp\{-\Delta_1 Q_1 - \Delta_2 Q_2\}$, where

$$Q_1 = \frac{1}{2a_2}\ln x + \frac{1}{4a_2}\ln(1+\lambda x^2) - \frac{1}{2a_2}\ln(1+2\lambda x^2), \tag{74}$$

$$Q_2 = \frac{a_0}{2a_2}\ln x + \frac{a_0}{4a_2}\ln(1+\lambda x^2) - \frac{a_0}{2a_2}\ln(1+2\lambda x^2) + \frac{1}{8\lambda}\ln(1+2\lambda x^2). \tag{75}$$

4 Conclusion and Discussion

In conclusion, we have obtained a new more general class of exactly solvable quantum nonlinear harmonic oscillators from the viewpoint of the raising and lowering operators. The energy spectrum for the QNHO and the ground state are given explicitly. It is also interesting and significant to study these new quantum nonlinear harmonic oscillators from the viewpoints of factorization method or the supersymmetric approach [10], which we shall investigate subsequently.

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